Published for SISSA by Springer

Received: *October 21, 2013*
Revised: *March 9, 2014*
Accepted: *March 11, 2014*
Published: *April 10, 2014*

The universal Racah-Wigner symbol for $U_q(osp(1|2))$

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Abstract: We propose a new and elegant formula for the Racah-Wigner symbol of self-dual continuous series of representations of $U_q(osp(1|2))$. It describes the entire fusing matrix for both NS and R sector of N=1 supersymmetric Liouville field theory. In the NS sector, our formula is related to an expression derived in [1]. Through analytic continuation in the spin variables, our universal expression reproduces known formulas for the Racah-Wigner coefficients of finite dimensional representations.

Keywords: Field Theories in Lower Dimensions, Quantum Groups, Conformal and W Symmetry

ArXiv ePrint: 1307.6866

Article funded by SCOAP^3.

doi:10.1007/JHEP04(2014)079

Contents

1 Introduction

The Racah-Wigner coefficients of Lie (super)algebras and their deformations play an important role in modern mathematical physics. Up to some normalization dependent prefactors, they coincide with the so-called fusing matrix of 2-dimensional Wess-Zumino-Novikov-Witten (WZNW) models and hence feature very prominently in the conformal bootstrap of these models and many descendants thereof. In fact, they do not only provide the coefficients in the bootstrap equations but also furnish some of their famous solutions e.g. for the bulk and boundary operator product coefficients. This dual purpose of the Racah-Wigner coefficients is based on a number of identities they satisfy, most importantly the well-known pentagon equation. The same identities are also exploited in the construction of state-sum models for topological 3-manifold invariants. These provide another important area in which Racah-Wigner symbols appear.

Recently, two of the authors and Leszek Hadasz constructed the Racah-Wigner symbol for a series of self-dual representations of $\mathrm{U}_q(\mathrm{osp}(1|2))$ [1] for $q = \exp(i\pi b^2)$ and real b^2. They also verified that the resulting expressions agree with the fusing matrix of N=1 Liouville field theory in the Neveu-Schwarz (NS) sector [2, 3]. A central goal of the present work is to extend the previous expression to include both NS and Ramond (R) sector fields. The way in which we shall achieve our goal is quite interesting in its own right.

Let us recall that the expression for the Racah-Wigner symbol found in [1] generalized previous formulas by Ponsot and Teschner for the Racah-Wigner symbol of $\mathrm{U}_q(\mathrm{sl}(2))$ [4, 5]. In a remarkable recent paper [6], Teschner and Vartanov found an alternative and much more natural way to express the same Racah-Wigner symbol. In particular, the new formulation is very closely modeled after the famous expressions for the Racah-Wigner coefficients of finite dimensional $\mathrm{U}_q(\mathrm{sl}(2))$ representations [7, 8], only that an integral appears instead of the usual summation and q-factorials are replaced by double Gamma functions.

Our strategy here is to extend the Teschner-Vartanov expressions for the Racah-Wigner symbol of $\mathrm{U}_q(\mathrm{sl}(2))$ to the supersymmetric case. Up to certain sign factors, this step is relatively straight-forward, taking into account some of the properties of the formula derived in [1]. The resulting expression is so natural that its extension to the R sector is rather easy to guess. Only the sign factors are a bit tricky to extend. We shall come up with a concrete proposal. In order to test our prescription for both NS and R sector labels we shall continue the integral formulas from spins $\alpha \in Q/2 + i\mathbb{R}$ to the discrete set $j = -\alpha/b \in \mathbb{N}/2$ at which the integrals can be evaluated by summing over certain residues.

When j is integer, the result of this evaluation gives the known 6J symbols for finite dimensional spin j representations of $\mathrm{U}_q(\mathrm{osp}(1|2))$ [9, 10]. This limit only uses information from the NS sector, but can be considered a very strong test of our proposal for the universal Racah-Wigner symbol, including the sign factors we prescribe in the NS sector.

In order to probe the R sector of the theory we make use of a remarkable observation in [11, 12]. These authors found that the 6J symbols for finite dimensional integer spin representations of $\mathrm{U}_{q'}(\mathrm{sl}(2))$ and $\mathrm{U}_q(\mathrm{osp}(1|2))$ actually coincide when $q' = i\sqrt{q}$. Because of the usual relation between the deformation parameter $q = \exp(i\pi/(2k+3)$ and the level k, the deformation parameter q' actually tends to $q' = i$ in the semiclassical limit $k \to \infty$ of $\mathrm{U}_q(\mathrm{osp}(1|2))$, i.e. it is associated to a point $q' = \exp(i\pi/(k+2)$ with $k = 0$, deeply in the quantum region of $\mathrm{U}_{q'}(\mathrm{sl}(2))$. In this sense, the numerical coincidences between 6J symbols of finite dimensional representations observed in [11, 12] can be thought of as a non-perturbative duality.[1] In our context we will find that the limiting $\mathrm{U}_q(\mathrm{osp}(1|2))$ Racah-Wigner symbols with discrete weights, including those corresponding to half-integer spin j, coincide with the 6J symbols of finite dimensional representations of $\mathrm{U}_{q'}(\mathrm{sl}(2))$. Thereby, we provide highly non-trivial evidence for our choice of sign factors in the R sector of the theory.

The tests of our proposal we described in the previous two paragraphs exhaust the data provided by finite dimensional representations of deformed universal enveloping algebras. On the other hand, we can evaluate our proposed Racah-Wigner symbol for a larger set of labels α which are parametrized by a pair of spin labels (j, j'). When $j' = 0$, we are back to the case discussed above. But for nontrivial values of j' the limiting value of the Racah-Wigner symbol may be written as a product of two 6J symbols with different values of q. In reaching such a conclusion, details of the sign factors become even more crucial. While the result has no direct interpretation in terms of finite dimensional representation theory of universal enveloping algebras, it can be understood from the relation between Liouville theory and minimal models in conformal field theory. Hence it adds quite significantly to the testing of our main proposal.

The plan of this paper is as follows. In the next section we shall re-address the case of $\mathrm{U}_q(\mathrm{sl}(2))$ and show how to recover the Racah-Wigner coefficients of finite dimensional representations from the formula of Teschner and Vartanov. After this warm-up, we can turn to the supersymmetric case in section 3. There we propose a new expression for the Racah-Wigner symbol of $\mathrm{U}_q(\mathrm{osp}(1|2))$. The comparison with the 6J symbols for integer spin representations of $\mathrm{U}_q(\mathrm{osp}(1|2))$ and with finite dimensional representations of $\mathrm{U}_q(\mathrm{sl}(2))$

[1]We thank Edward Witten for stressing this aspect of the duality in a private conversation.

is performed in section 4. We conclude this work with a number of comments on open problems, including some speculations about the extension of the duality between $\mathrm{U}_q(\mathrm{sl}(2))$ and $\mathrm{U}_q(\mathrm{osp}(1|2))$ to infinite dimensional self-dual representations.

2 The Racah-Wigner symbol of $\mathrm{U}_q(\mathrm{sl}(2))$

In this section we will start from a recent integral formula for the Racah-Wigner symbol of a self-dual series of representations of $\mathrm{U}_q(\mathrm{sl}(2))$ with $q = e^{i\pi b^2}$, parametrized by $\alpha = Q/2 + i\mathbb{R}, Q = b + b^{-1}$ [6]. This symbol turns out to simplify when we consider its analytic continuation to parameters $\alpha = -jb - j'b^{-1};\ j, j' \in \frac{\mathbb{N}}{2}$. In fact, it can be then written as a sum over finitely many pole contributions. We can compare the resulting expressions with the formulas for Racah-Wigner coefficients of finite dimensional representations of $\mathrm{U}_q(\mathrm{sl}(2))$ and find complete agreement, at least up to some normalization dependent prefactors.

Let us begin our discussion by reviewing the formulas for the universal Racah-Wigner coefficients of $\mathrm{U}_q(\mathrm{sl}(2))$ which were proposed by Teschner and Vartanov [6]

$$\begin{Bmatrix} \alpha_1 & \alpha_3 & \alpha_s \\ \alpha_2 & \alpha_4 & \alpha_t \end{Bmatrix} = \Delta(\alpha_1,\alpha_2,\alpha_s)\Delta(\alpha_s,\alpha_3,\alpha_4)\Delta(\alpha_t,\alpha_3,\alpha_2)\Delta(\alpha_4,\alpha_t,\alpha_1) \qquad (2.1)$$

$$\times \int_{\mathcal{C}} du\, \frac{S_b(u-\alpha_{12s})\,S_b(u-\alpha_{s34})\,S_b(u-\alpha_{23t})\,S_b(u-\alpha_{1t4})}{S_b(\alpha_{1234}-u)\,S_b(\alpha_{st13}-u)\,S_b(\alpha_{st24}-u)\,S_b(2Q-u)}$$

where

$$\Delta(\alpha_3,\alpha_2,\alpha_1) = \left(\frac{S_b(\alpha_{123}-Q)}{S_b(\alpha_{12}-\alpha_3)\,S_b(\alpha_{23}-\alpha_1)\,S_b(\alpha_{31}-\alpha_2)}\right)^{\frac{1}{2}} \qquad (2.2)$$

and the multi-index of α denotes summation, e.g. $\alpha_{ij} = \alpha_i + \alpha_j$. The integral is defined for $\alpha_j = Q/2 + i\mathbb{R}, Q = b + b^{-1}$ by a contour $\mathcal{C}$ which crosses the real axis in the interval $(\frac{3Q}{2}, 2Q)$ and approaches $2Q + i\mathbb{R}$ near infinity. The double sine function $S_b(x)$ is given in terms of Barnes' double Gamma function. Its definition and some relevant properties are listed in appendix A. Let us note that Teschner and Vartanov were able to show that the expression (2.1) agrees with an earlier formula for the Racah-Wigner symbol of $\mathrm{U}_q(\mathrm{sl}(2))$ that was established by Teschner and Ponsot [4, 5]. Thus the Racah-Wigner symbol (2.1) coincides with the fusion matrix of Liouville theory [6, 13]. Because of this relation with conformal field theory (CFT) we shall use some CFT terminology from time to time. In particular, we will refer to the labels α_i, $i = 1,\ldots,4$ and α_s, α_t as *external* and *intermediate* parameters, respectively.

Let us begin our analysis of the Racah-Wigner symbols (2.1) with the prefactor of the integral in the first line. Insertion of the definition (2.2) gives

$$\mathcal{P}(\alpha_i) \equiv \Delta(\alpha_1,\alpha_2,\alpha_s)\Delta(\alpha_s,\alpha_3,\alpha_4)\Delta(\alpha_t,\alpha_3,\alpha_2)\Delta(\alpha_4,\alpha_t,\alpha_1) = \qquad (2.3)$$

$$\left(\frac{S_b(\alpha_{12s}-Q)S_b(\alpha_{s34}-Q)}{S_b(\alpha_{12}-\alpha_s)\,S_b(\alpha_{2s}-\alpha_1)\,S_b(\alpha_{1s}-\alpha_2)\,S_b(\alpha_{34}-\alpha_s)\,S_b(\alpha_{3s}-\alpha_4)\,S_b(\alpha_{4s}-\alpha_3)}\right)^{\frac{1}{2}}$$

$$\times\left(\frac{S_b(\alpha_{23t}-Q)S_b(\alpha_{1t4}-Q)}{S_b(\alpha_{23}-\alpha_t)\,S_b(\alpha_{2t}-\alpha_3)\,S_b(\alpha_{3t}-\alpha_2)\,S_b(\alpha_{14}-\alpha_t)\,S_b(\alpha_{1t}-\alpha_4)\,S_b(\alpha_{4t}-\alpha_1)}\right)^{\frac{1}{2}}.$$

We observe that the prefactor vanishes each time one of the external parameters α_i approaches the so called degenerate value $\alpha_{n,n'} \equiv -\frac{nb}{2} - \frac{n'}{2b}$; $n, n' \in \mathbb{Z}_{\geq 0}$, and one of the intermediate parameters $\alpha_x, (x = s, t)$ satisfies the condition

$$\alpha_x = \alpha_j - \frac{xb}{2} - \frac{x'}{2b}\,, \quad x \in \{-n, -n+2, \dots, n\}\,, \quad x' \in \{-n', -n'+2, \dots, n'\} \tag{2.4}$$

where the labels $i, j \in \{1,2\}$ or $\{3,4\}$ for $x = s$, and $i, j \in \{2,3\}$ or $\{1,4\}$ for $x = t$. In Liouville theory, the values $\alpha_{n,n'}$ are associated with so-called degenerate fields which satisfy additional null vector decoupling equations. These restrict the possible operator products to a finite set of terms which are labeled by parameters satisfying so-called *fusion rules*, i.e. conditions of the form (2.4).

Let us now consider a limit of the Racah-Wigner symbol where one of the external parameters becomes degenerate and the intermediate parameter α_s satisfies the condition (2.4). As we shall show below, the limit is finite and non-zero because the integral in eq. (2.1) contributes singular terms canceling zeroes from the prefactor. In order to see how this works in detail, let us focus on the limit $\alpha_2 \to -\frac{nb}{2}$ $(n > 0)$ and $\alpha_s \to \alpha_1 - \frac{sb}{2}$. The zero in the prefactor comes from the first two terms in the denominator of eq. (2.3)

$$\begin{aligned}\lim_{\substack{\alpha_2 \to -\frac{nb}{2} \\ \alpha_s \to \alpha_1 - \frac{sb}{2}}} \left(S_b(\alpha_{12} - \alpha_s)\, S_b(\alpha_{2s} - \alpha_1)\right)^{-\frac{1}{2}} &= \left(S_b\left(\frac{s-n}{2}b\right) S_b\left(-\frac{s+n}{2}b\right)\right)^{-\frac{1}{2}} \\ &= \left(-2\sin(\pi b^2)\right)^{\frac{n}{2}} \left(\left[\frac{n-s}{2}\right]! \left[\frac{n+s}{2}\right]!\right)^{\frac{1}{2}} S_b(0)^{-1}\end{aligned}$$

where we used the shift relation (A.4) for the double sine function and the notation

$$[x] = \frac{\sin(\pi b^2 x)}{\sin \pi b^2}\,. \tag{2.5}$$

For integer x the factorial $[x]!$ is defined as,

$$[x]! = \prod_{a=1}^{x} [a] = \left(\sin \pi b^2\right)^{-x} \prod_{a=1}^{x} \sin(\pi b^2 a)\,. \tag{2.6}$$

In order to obtain a finite non-zero limit for the full Racah-Wigner symbol, the integral must contribute a divergent factor $S_b(0)$ to cancel the corresponding term from the prefactor. Let us therefore take a closer look at the integral in eq. (2.1). Its analytic continuation to $\alpha_2 = -\frac{nb}{2}, \alpha_s = \alpha_1 - \frac{sb}{2}$ is defined by the same integral with a deformed contour $\mathcal{C}'$, see figure 1 and figure 2 for the cases $s \geq 0$ and $s < 0$, respectively. As we deform the original contour we have to take into account contributions from poles. We shall split these into two groups and denote them by I_1, I_2, respectively,

$$\begin{aligned}&\int_{\mathcal{C}'} du S_b(u - \alpha_{12s})\, S_b(u - \alpha_{s34})\, S_b(u - \alpha_{23t})\, S_b(u - \alpha_{1t4}) \\ &\qquad S_b(\alpha_{1234} - u)\, S_b(\alpha_{st13} - u)\, S_b(\alpha_{st24} - u)\, S_b(2Q - u) = I_{\text{reg}} + I_1 + I_2\,.\end{aligned} \tag{2.7}$$

The first term I_{reg} denotes the integral over the original contour and a regular contribution. The singular terms I_1 and I_2 will be described and calculated in the next few paragraphs.

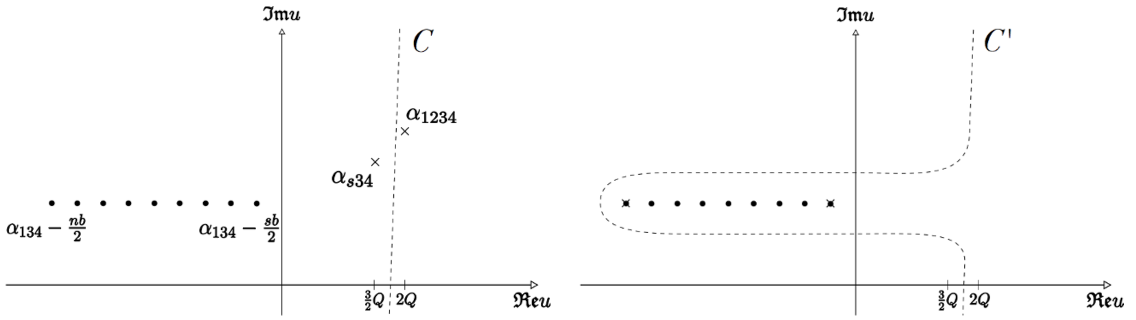

Figure 1. The original integration contour $\mathcal{C}$ passes between the points $u = \alpha_{s34}$ and $u = \alpha_{1234}$. As we deform the contour to $\mathcal{C}'$, the poles contribute to singular term I_1 due to the pinching mechanism.

By definition, the first singular term I_1 has origin in the two double sine functions $S_b(u - \alpha_{s34})\, S_b(\alpha_{1234} - u)$. Let us first consider the case of $s \geq 0$. Then the poles of $S_b(u - \alpha_{s34})$ in $u = \alpha_{s34} - pb$ $(0 \leq p \leq \frac{n-s}{2})$ lie on the left side of the contour $\mathcal{C}$, see figure 1. When we deform the contour to $\mathcal{C}'$ we thus obtain contributions from non-vanishing residues in these points. These residues are proportional to the other double sine function $S_b(\alpha_{1234} - \alpha_{s34} + pb)$ and in the limit $\alpha_2 \to -\frac{nb}{2}, \alpha_s \to \alpha_1 - \frac{sb}{2}$ become singular. This is the so called pinching mechanism, see e.g. [5], Lemma 3 and [2, 14] for similar calculations. In the end we obtain the following sum

$$
\begin{aligned}
I_1 = \sum_{p=0}^{\frac{n-s}{2}} \Bigg(& \frac{\left(-2\sin(\pi b^2)\right)^{\frac{s-n}{2}} S_b(0)}{[p]!\,\left[\frac{n-s}{2} - p\right]!} S_b\left(\alpha_{34} - \alpha_1 + \frac{nb}{2} - pb\right) \\
& S_b(\alpha_{1t} - \alpha_4 + pb) S_b\left(\alpha_{14} - \alpha_t + \frac{(n-s)b}{2} - pb\right) S_b\left(\alpha_3 - \alpha_t - \frac{sb}{2} - pb\right) \\
& S_b(\alpha_t - \alpha_3 - \frac{nb}{2} + pb) S_b\left(2Q - \alpha_{134} + \frac{sb}{2} + pb\right)\Bigg).
\end{aligned}
\tag{2.8}
$$

When $s < 0$ the function $S_b(u - \alpha_{s34})$ has poles in $u = \alpha_{s34} - pb$ $(-\frac{s}{2} \leq p \leq \frac{n-s}{2})$. In the limit $\alpha_s \to \alpha_1 - \frac{sb}{2}$ these are situated on the left side of the contour $\mathcal{C}$, see figure 2. On the other hand the function $S_b(\alpha_{1234} - u)$ has poles in $u = \alpha_{1234} + pb$ $(0 \leq p \leq -\frac{s}{2})$ that are located on the right side of the contour. While deforming the contour to $\mathcal{C}'$ we pick up contributions from all these poles. Each residue is proportional to $S_b(\alpha_{12} - \alpha_s + pb)$ and develops a singularity in the limit $\alpha_2 \to -\frac{nb}{2}, \alpha_s \to \alpha_1 - \frac{sb}{2}$. The final result will be the same as in the case (2.8) where we assumed $s \geq 0$.

The term we have denoted by I_2 come from the poles of the function $S_b(u - \alpha_{1t4})$ in $u = \alpha_{1t4} - p'b$ for $0 \leq p' \leq \frac{n+s}{2}$. Since $s > -n$, the poles lie on the left side of the contour $\mathcal{C}$, independently of the sign of the parameter s (analogous to figure 1). The residues of all poles we pass while deforming the contour are proportional to $S_b(\alpha_{st24} - \alpha_{1t4} + p'b)$. In

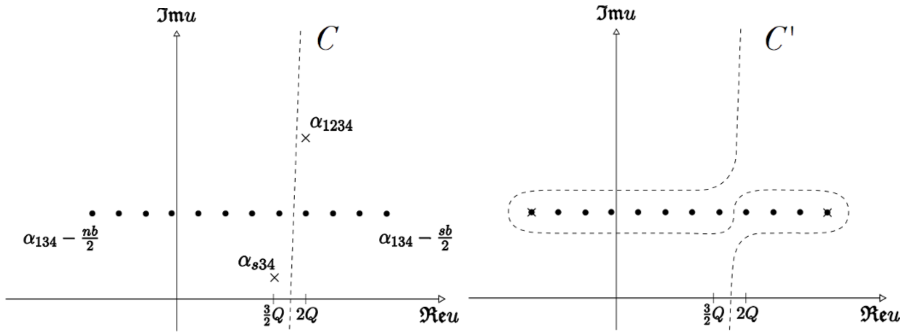

Figure 2. When $s<0$ we have to deform the contour in the above way. The poles appear on both sides of the contour $\mathcal{C}$ and they all give singular contribution to I_1.

the limit $\alpha_2 \to -\frac{nb}{2}, \alpha_s \to \alpha_1 - \frac{sb}{2}$ they contribute to the second sum of singular terms,

$$\begin{aligned} I_2 = \sum_{p'=0}^{\frac{n+s}{2}} \Bigg(& \frac{\left(-2\sin(\pi b^2)\right)^{-\frac{n+s}{2}} S_b(0)}{[p']!\left[\frac{n+s}{2}-p'\right]!} S_b\left(\alpha_{t4}-\alpha_1+\frac{(s+n)b}{2}-p'b\right) \\ & S_b\left(\alpha_{14}-\alpha_3+\frac{nb}{2}-p'b\right) S_b\left(\alpha_t-\alpha_3+\frac{sb}{2}-p'b\right) S_b(2Q-\alpha_{1t4}+p'b) \\ & S_b\left(\alpha_3-\alpha_t-\frac{nb}{2}+p'b\right) S_b\left(\alpha_{13}-\alpha_4-\frac{sb}{2}+p'b\right)\Bigg). \end{aligned} \tag{2.9}$$

Combining the two divergent terms I_1, I_2 given in eqs. (2.8), (2.9) with the prefactor $\mathcal{P}(\alpha_i)$ from eq. (2.3) we obtain a finite result for the limit,

$$\lim_{\substack{\alpha_2\to-\frac{nb}{2}\\ \alpha_s\to\alpha_1-\frac{sb}{2}}} \begin{Bmatrix} \alpha_1 & \alpha_3 & \alpha_s \\ \alpha_2 & \alpha_4 & \alpha_t \end{Bmatrix} = \lim_{\substack{\alpha_2\to-\frac{nb}{2}\\ \alpha_s\to\alpha_1-\frac{sb}{2}}} \mathcal{P}(\alpha_i)\,(I_1+I_2) \tag{2.10}$$

$$\begin{aligned} = & \left(\frac{S_b(\alpha_{14}+\alpha_t-Q)S_b(\alpha_3+\alpha_t-\frac{nb}{2}-Q)}{S_b\left(\alpha_3-\alpha_t-\frac{nb}{2}\right)S_b\left(\alpha_t-\alpha_3-\frac{nb}{2}\right)S_b(\alpha_{14}-\alpha_t)\,S_b(\alpha_{1t}-\alpha_4)\,S_b(\alpha_{4t}-\alpha_1)}\right)^{\frac{1}{2}} \\ & \left(\frac{\left[\frac{n-s}{2}\right]!\left[\frac{n+s}{2}\right]! S_b\left(2\alpha_1-\frac{(s+n)b}{2}-Q\right) S_b\left(2\alpha_{134}-\frac{sb}{2}-Q\right)}{S_b\left(\alpha_{3t}+\frac{nb}{2}\right) S_b\left(2\alpha_1+\frac{(n-s)b}{2}\right) S_b\left(\alpha_{34}-\alpha_1+\frac{sb}{2}\right) S_b\left(\alpha_{13}-\alpha_4-\frac{sb}{2}\right) S_b\left(\alpha_{14}-\alpha_3-\frac{sb}{2}\right)}\right)^{\frac{1}{2}} \\ & \Bigg\{\sum_{q=0}^{\frac{n-s}{2}} \frac{\left(-2\sin(\pi b^2)\right)^{\frac{s}{2}}}{[q]!\left[\frac{n-s}{2}-q\right]!} S_b(\alpha_{34}-\alpha_1+\frac{nb}{2}-qb)\,S_b(\alpha_{14}-\alpha_t+\frac{(n-s)b}{2}-qb) \\ & \quad S_b\left(\alpha_3-\alpha_t-\frac{sb}{2}-qb\right)S_b\left(\alpha_t-\alpha_3-\frac{nb}{2}+qb\right)S_b(\alpha_{1t}-\alpha_4+qb)S_b\left(2Q-\alpha_{134}+\frac{sb}{2}+qb\right) \\ & \quad +\sum_{p'=0}^{\frac{n+s}{2}} \frac{\left(-2\sin(\pi b^2)\right)^{-\frac{s}{2}}}{[p']!\left[\frac{n+s}{2}-p'\right]!} S_b\left(\alpha_{t4}-\alpha_1+\frac{(s+n)b}{2}-p'b\right) S_b\left(\alpha_{14}-\alpha_3+\frac{nb}{2}-p'b\right) \\ & \quad S_b\left(\alpha_t-\alpha_3+\frac{sb}{2}-p'b\right)S_b\left(\alpha_3-\alpha_t-\frac{nb}{2}+p'b\right)S_b\left(\alpha_{13}-\alpha_4-\frac{sb}{2}+p'b\right)S_b(2Q-\alpha_{1t4}+p'b)\Bigg\}. \end{aligned}$$

Suppose now that the other intermediate parameter α_t also satisfies condition (2.4) i.e. $\alpha_t \to \alpha_3 - \frac{tb}{2}$. Then the prefactor in the formula above gives zero. On the other hand in each term of the sums there are double poles for $t \in \{-n+2p, -n+2p+2, \ldots, s+2p\}$ and $t \in \{s-2p', s-2p'+2, \ldots, n-2p'\}$ coming from $S_b(\alpha_3 - \alpha_t - \frac{sb}{2} - pb)\, S_b(\alpha_t - \alpha_3 - \frac{nb}{2} + pb)$ and $S_b(\alpha_t - \alpha_3 - p'b + \frac{sb}{2})\, S_b(\alpha_3 - \alpha_t + \alpha_2 - p'b)$, respectively. The residue for a given $\alpha_t \to \alpha_3 - \frac{tb}{2}$ takes the form

$$\begin{aligned}
&\underset{\alpha_t \to \alpha_3 - \frac{tb}{2}}{Res}\left(\lim_{\substack{\alpha_2 \to -\frac{nb}{2} \\ \alpha_s \to \alpha_1 - \frac{sb}{2}}} \begin{Bmatrix} \alpha_1 & \alpha_3 & \alpha_s \\ \alpha_2 & \alpha_4 & \alpha_t \end{Bmatrix}\right) = \left(\frac{S_b(2\alpha_1 - \frac{(s+n)b}{2} - Q) S_b(2\alpha_3 - \frac{(t+n)b}{2} - Q)}{S_b(2\alpha_1 + \frac{(n-s)b}{2})\, S_b(2\alpha_3 + \frac{(n-t)b}{2})}\right)^{\frac{1}{2}} \\
&\sum_{p=\max\{0,\frac{t-s}{2}\}}^{\min\{\frac{n-s}{2},\frac{n+t}{2}\}} \frac{2\left([\frac{n-s}{2}]!\, [\frac{n+s}{2}]!\, [\frac{n-t}{2}]!\, [\frac{n+t}{2}]!\right)^{\frac{1}{2}}}{[p]!\, [\frac{n-s}{2} - p]!\, [\frac{s-t}{2} + p]!\, [\frac{n+t}{2} - p]!} \frac{S_b\left(\alpha_{13} - \alpha_4 + pb - \frac{tb}{2}\right)}{\left(S_b\left(\alpha_{13} - \alpha_4 - \frac{sb}{2}\right) S_b\left(\alpha_{13} - \alpha_4 - \frac{tb}{2}\right)\right)^{\frac{1}{2}}} \\
&\quad \frac{S_b\left(\alpha_{34} - \alpha_1 - pb + \frac{nb}{2}\right)}{\left(S_b\left(\alpha_{34} - \alpha_1 + \frac{sb}{2}\right) S_b\left(\alpha_{34} - \alpha_1 - \frac{tb}{2}\right)\right)^{\frac{1}{2}}} \frac{S_b\left(\alpha_{14} - \alpha_3 - pb + \frac{(n+t-s)b}{2}\right)}{\left(S_b\left(\alpha_{14} - \alpha_3 - \frac{sb}{2}\right) S_b\left(\alpha_{14} - \alpha_3 + \frac{tb}{2}\right)\right)^{\frac{1}{2}}} \\
&\quad \frac{\left(S_b\left(\alpha_{134} - \frac{sb}{2} - Q\right) S_b\left(\alpha_{134} - \frac{tb}{2} - Q\right)\right)^{\frac{1}{2}}}{S_b\left(\alpha_{134} - \frac{sb}{2} - pb - Q\right)}
\end{aligned} \tag{2.11}$$

where we redefined the second summation parameter $p' = p - \frac{t-s}{2}$ in order to obtain two identical sums. Let us denote the residue above as

$$\begin{Bmatrix} \alpha_1 & \alpha_3 & \alpha_1 - \frac{sb}{2} \\ -\frac{nb}{2} & \alpha_4 & \alpha_3 - \frac{tb}{2} \end{Bmatrix}' \equiv \underset{\alpha_t \to \alpha_3 - \frac{tb}{2}}{Res}\left(\lim_{\substack{\alpha_2 \to -\frac{nb}{2} \\ \alpha_s \to \alpha_1 - \frac{sb}{2}}} \begin{Bmatrix} \alpha_1 & \alpha_3 & \alpha_s \\ \alpha_2 & \alpha_4 & \alpha_t \end{Bmatrix}\right). \tag{2.12}$$

Now one can set all the other external parameters α_i $(i = 1, 3, 4)$ to degenerate values, $\alpha_i \to -j_i b$, $2j_i \in \mathbb{Z}_{\geq 0}$. In this case, eq. (2.11) takes the form

$$\begin{aligned}
&\begin{Bmatrix} -j_1 b & -j_3 b & -j_1 b - \frac{sb}{2} \\ -\frac{nb}{2} & -j_4 b & -j_3 b - \frac{t}{2} b \end{Bmatrix}' = 2\left(\frac{[2j_1 + \frac{s-n}{2}]!}{[2j_1 + \frac{n+s}{2} + 1]!} \frac{[2j_3 + \frac{t-n}{2}]!}{[2j_3 + \frac{n+t}{2} + 1]!}\right)^{\frac{1}{2}} \\
&\quad \sum_{p=\max\{0,\frac{t-s}{2}\}}^{\min\{\frac{n-s}{2},\frac{t+n}{2}\}} (-1)^{j_1 + j_3 - p + \frac{n+t}{2}} \frac{\left([\frac{n-s}{2}]!\, [\frac{n+s}{2}]!\, [\frac{n-t}{2}]!\, [\frac{n+t}{2}]!\right)^{\frac{1}{2}}}{[p]!\, [p + \frac{s-t}{2}]!\, [\frac{n-s}{2} - p]!\, [\frac{n+t}{2} - p]!} \\
&\qquad \frac{[j_{134} + p + \frac{s}{2} + 1]!}{\left([j_{134} + \frac{s}{2} + 1]!\, [j_{134} + \frac{t}{2} + 1]!\right)^{\frac{1}{2}}} \frac{\left([j_{13} - j_4 + \frac{s}{2}]!\, [j_{13} - j_4 + \frac{t}{2}]!\right)^{\frac{1}{2}}}{[j_{13} - j_4 - p + \frac{t}{2}]!} \\
&\qquad \frac{\left([j_{34} - j_1 - \frac{s}{2}]!\, [j_{34} - j_1 + \frac{t}{2}]!\right)^{\frac{1}{2}}}{[j_{34} - j_1 + p - \frac{n}{2}]!} \frac{\left([j_{14} - j_3 + \frac{s}{2}]!\, [j_{14} - j_3 - \frac{t}{2}]!\right)^{\frac{1}{2}}}{[j_{14} - j_3 + p - \frac{t+n-s}{2}]!}
\end{aligned}$$

where we assumed that $\frac{n}{2} - \frac{\alpha_{134}}{b} = j_{134} + \frac{n}{2} \in \mathbb{N}$ and we expressed the S_b functions in terms of the [.]-factorials (2.6). The minus sign under the sum comes from the difference in the shift relations (A.4) concerning $S_b(-xb)$ and $S_b(-xb+Q)$. Denoting $j_2 = \frac{n}{2}, j_s = j_1 + \frac{s}{2}, j_t = j_3 + \frac{t}{2}$ and shifting the summation parameter to $z = p + j_{s34}$, one can see our limit coincides with the 6J symbol for finite dimensional representations of the quantum deformed algebra $\mathrm{U}_q(\mathrm{sl}(2))$,

$$\begin{Bmatrix} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{Bmatrix}' = \frac{(-1)^{j_s+j_t}([2j_s+1]_q[2j_t+1]_q)^{-\frac{1}{2}}}{2\sin(\pi b^2)\sin(-\pi b^{-2})} \begin{pmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{pmatrix}_q \tag{2.13}$$

where the deformation parameter q is given in terms of b as $q = e^{i\pi b^2}$ and the quantum numbers $[.]_q$ of $\mathrm{U}_q(\mathrm{sl}(2))$ are equal those defined in eq. (2.5), i.e.

$$[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} = [x] \; . \tag{2.14}$$

Thus we conclude that the residue of the Racah-Wigner coefficient (2.12) analytically continued to $\alpha_i = -j_i b$, $2j_i \in \mathbb{Z}_{\geq 0}$ is equivalent to the 6J symbol of the finite dimensional representations of the quantum deformed algebra $\mathrm{U}_q(\mathrm{sl}(2))$.

The 6J symbol of finite dimensional representations of $\mathrm{U}_q(\mathrm{sl}(2))$ is given by the following sum [7, 8, 15]

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{pmatrix}_q &= \sqrt{[2j_s+1]_q[2j_t+1]_q}\,(-1)^{j_{12}-j_{34}-2j_s} \\ &\times \sum_{z\geq 0} \frac{(-1)^z\, \Delta_q(j_s,j_2,j_1)\Delta_q(j_s,j_3,j_4)\Delta_q(j_t,j_3,j_2)\Delta_q(j_4,j_t,j_1)\,[z+1]_q!}{[z-j_{12s}]_q!\,[z-j_{34s}]_q!\,[z-j_{14t}]_q!\,[z-j_{23t}]_q![j_{1234}-z]_q!\,[j_{13st}-z]_q!\,[j_{24st}-z]_q!} \,. \end{aligned} \tag{2.15}$$

Here, the summation extend over those values of z for which all arguments of the quantum number $[.]_q$ are non-negative. In addition we used the shorthand

$$\Delta_q(a,b,c) = \sqrt{[-a+b+c]_q!\,[a-b+c]_q!\,[a+b-c]_q!/[a+b+c+1]_q!}\,.$$

It is worth pointing out the similarities between the expressions (2.15) and the original formula (2.1). In passing to eq. (2.15), the four factors Δ got replaced by Δ_q while the eight functions S_b have contributed the same number of quantum factorials. In addition, the integration over u became a summation over z.

In the above calculation we have restricted α to a subset of degenerate labels $\alpha = -jb - j'b^{-1}$ with $j' = 0$. One may certainly wonder about the more general case with $j' \neq 0$. It turns out that the corresponding limit of the Racah-Wigner symbol can still be evaluated using pretty much the same steps as before. More precisely, we can continue the Racah-Wigner symbol (2.1) to general degenerate values

$$\alpha_i \to -j_i b - j_i' b^{-1}; \quad j, j' \in \frac{\mathbb{Z}_{\geq 0}}{2}, \tag{2.16}$$

evaluate the residue at $\alpha_t = \alpha_j - \frac{t}{2}b - \frac{t'}{2}b^{-1}$ and restrict the other intermediate parameter α_s to the values (2.4). These steps define the symbol

$$\begin{Bmatrix} -j_1 b - j_1' b^{-1} & -j_3 b - j_3' b^{-1} & -j_s b - j_s' b^{-1} \\ -j_2 b - j_2' b^{-1} & -j_4 b - j_4' b^{-1} & -j_t b - j_t' b^{-1} \end{Bmatrix}' \tag{2.17}$$
$$\equiv \lim_{\substack{\alpha_j \to -j_j b - j_j' b^{-1} \\ \alpha_k \to -j_k b - j_k' b^{-1} \\ \alpha_l \to -j_l b - j_l' b^{-1}}} \; \underset{\alpha_t \to \alpha_j - \frac{t}{2}b - \frac{t'}{2}b^{-1}}{Res} \left(\lim_{\substack{\alpha_i \to -j_i b - j_i' b^{-1} \\ \alpha_s \to \alpha_k - \frac{s}{2}b - \frac{s'}{2}b^{-1}}} \begin{Bmatrix} \alpha_1 & \alpha_3 & \alpha_s \\ \alpha_2 & \alpha_4 & \alpha_t \end{Bmatrix} \right),$$

where

$$j_s = j_k + \frac{s}{2}, \quad j_s' = j_k' + \frac{s'}{2}; \qquad j_t = j_j + \frac{t}{2}, \quad j_t' = j_j' + \frac{t'}{2}.$$

Using the properties of double sine functions (A.5) and the assumption

$$j_{1234}, \; j_{1234}' \in \mathbb{Z}_{\geq 0},$$

one can express the limit as a product of two 6J symbols of finite dimensional representations of the quantum deformed algebra $\mathrm{U}_q(\mathrm{sl}(2))$

$$\begin{Bmatrix} -j_1 b - j_1' b^{-1} & -j_3 b - j_3' b^{-1} & -j_s b - j_s' b^{-1} \\ -j_2 b - j_2' b^{-1} & -j_4 b - j_4' b^{-1} & -j_t b - j_t' b^{-1} \end{Bmatrix}' = (-1)^{j_{st} + j_{st}' + 3 j_{1234st} j_{1234st}' - j_{13} j_{13}' - j_{24} j_{24}' - j_{st} j_{st}'}$$
$$\times \frac{([2j_s+1]_q [2j_t+1]_q [2j_s'+1]_{q'} [2j_t'+1]_{q'})^{-\frac{1}{2}}}{2\sin(\pi b^2)\sin(-\pi b^{-2})} \begin{pmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{pmatrix}_q \begin{pmatrix} j_1' & j_2' & j_s' \\ j_3' & j_4' & j_t' \end{pmatrix}_{q'}, \tag{2.18}$$

where the deformation parameters assume two different values, namely $q = e^{i\pi b^2}$ and $q' = e^{i\pi b^{-2}}$.

As we anticipated in the introduction, the result has an interesting CFT interpretation. The limit we consider gives the value of the fusion matrix in Liouville theory where all representations are degenerate and both intermediate representations satisfy the fusion rules. The resulting numbers are expected to describe the fusing matrix of Virasoro minimal models, at least after continuation of the parameter b to the imaginary discrete values $b = i\beta$ with $\beta^2 = \frac{m+1}{m}$. The associated central charges

$$c = 13 + 6(b^2 + b^{-2}) \;\to\; 13 - 6(\beta^2 + \beta^{-2}),$$

take discrete values with $c < 1$. When parametrized in terms of the integer m, our parameters q and q' read

$$q = e^{-i\pi\beta^2} = e^{-i\pi\frac{m+1}{m}}, \qquad q' = e^{-i\pi\beta^{-2}} = e^{-i\pi\frac{m}{m+1}}.$$

Since $\mathrm{U}_q(\mathrm{sl}(2))$ 6J symbols are invariant with respect to $q \to q^{-1}$, we can also use the parameters $q_1 = \exp(i\pi\frac{m+1}{m})$ and $q_2 = \exp(i\pi\frac{m}{m+1})$ on the right hand side of eq. (2.18). The result agrees then with the fusing matrix of (unitary) minimal models [16–18].[2] Thus

[2] Often $\mathrm{U}_q(\mathrm{sl}(2))$ deformation parameters are defined as $q = e^{2i\pi\beta^{\pm 2}}$ which in our notation is equal to q_1^2, q_2^2.

we have shown that one can recover the fusion matrix of minimal models from the Racah-Wigner symbol (2.1).

Given the connection with minimal models, the product structure of our result (2.18) is easily understood from the famous coset construction,

$$\mathrm{MM}_k = (\mathrm{SU}(2)_k \times \mathrm{SU}(2)_1)/\mathrm{SU}(2)_{k+1}\,,$$

for Virasoro minimal models. Here the parameter k is related to $m = k+2$ by a finite shift. Sectors of the coset theory are labeled by three integers $(2j, 2j', 2l)$ where $0 \le 2j \le k$, $0 \le 2j' \le k+1$, $l = 0, \frac{1}{2}$. The last label does not play a role because it can be set to $l = 0$ using the so-called field identification symmetry. The two nontrivial factors in the fusing matrix are associated with the SU(2) Wess-Zumino-Witten (WZW) models at level k and $k+1$. While the $\mathrm{SU}(2)_k$ model contributes a factor with $\exp(2\pi i/(k+2)) = q_1^2$, the 6J symbol with $\exp(2\pi i/(k+3)) = q_2^2$ comes from the SU(2) WZW model at level $k+1$.

3 The supersymmetric Racah-Wigner symbol

After our warmup with the Racah-Wigner symbol of the $\mathrm{U}_q(\mathrm{sl}(2))$, we are now prepared to study its extension to the supersymmetric case. We shall define the supersymmetric Racah-Wigner symbol in the next few paragraphs and comment a bit on its relation with N=1 Liouville field theory and the Racah-Wigner symbol for self-dual representations of $\mathrm{U}_q(\mathrm{osp}(1|2))$. Then we perform an analysis along the lines of section 2, i.e. we compute the limit of the Racah-Wigner symbol for a discrete set of representation labels. The interpretation of the results is a bit more subtle than in the example of $\mathrm{U}_q(\mathrm{sl}(2))$. It has to wait until section 4.

As a supersymmetric extension of the Racah-Wigner symbol (2.1) we propose the following integral formula

$$\begin{aligned}\left\{\begin{matrix}\alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s}\\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t}\end{matrix}\right\}_{\nu_1\nu_2}^{\nu_3\nu_4} &= \delta_{\sum_i \nu_i = a_s+a_t \, mod\, 2}\Delta_{\nu_4}(\alpha_s,\alpha_2,\alpha_1)\Delta_{\nu_3}(\alpha_s,\alpha_3,\alpha_4)\Delta_{\nu_2}(\alpha_t,\alpha_3,\alpha_2)\\ \times\Delta_{\nu_1}(\alpha_4,\alpha_t,\alpha_1)\int_{\mathcal{C}} du \sum_{\nu=0}^{1}\Big((-1)^X & S_{1+\nu+\nu_4+a_s}(u-\alpha_{12s})\, S_{1+\nu+\nu_3+a_s}(u-\alpha_{s34})\\ & S_{1+\nu+\nu_2+a_t}(u-\alpha_{23t})\, S_{1+\nu+\nu_1+a_t}(u-\alpha_{1t4})\, S_{\nu+\nu_1+\nu_2+a_t}(\alpha_{1234}-u)\\ & S_{\nu+\nu_1+\nu_3+a_2}(\alpha_{st13}-u)\, S_{\nu+\nu_1+\nu_4+a_3}(\alpha_{st24}-u)\, S_\nu(2Q-u)\Big)\end{aligned} \tag{3.1}$$

where

$$\Delta_\nu(\alpha_3,\alpha_2,\alpha_1) = \left(\frac{S_{\nu+\frac{1}{2}a_{123}}(\alpha_{123}-Q)}{S_{\nu+\frac{1}{2}(a_{12}-a_3)}(\alpha_{12}-\alpha_3)S_{\nu+\frac{1}{2}(a_{23}-a_1)}(\alpha_{23}-\alpha_1)S_{\nu+\frac{1}{2}(a_{31}-a_2)}(\alpha_{31}-\alpha_2)}\right)^{\frac{1}{2}}$$

and the contour $\mathcal{C}$, as in the bosonic case, crosses the real axis in the interval $(\frac{3Q}{2}, 2Q)$ and approaches $2Q + i\mathbb{R}$ near infinity. Note that the arguments α^a of the Racah-Wigner symbol contain a continuous quantum number $\alpha \in Q/2 + i\mathbb{R}$ along with a superscript a that can take the values $a = 0$ and $a = 1$. The discrete label a keeps track on whether the corresponding representation is taken from the Neveu-Schwarz (NS) or Ramond (R) sector, respectively. We will comment a bit more on this below. We define the Racah-Wigner symbol for the discrete labels a_i satisfying the following conditions

$$a_s = a_1 + a_2 = a_3 + a_4 \, mod \, 2, \qquad a_t = a_1 + a_4 = a_2 + a_3 \, mod \, 2, \qquad \sum_{i=1}^{4} a_i = 0 \, mod \, 2, \quad (3.2)$$

otherwise the symbol is set to zero. The sign factor

$$(-1)^X = (-1)^{\nu(a_s\nu_1 + a_1\nu_3 + a_4\nu_4 + a_1 a_s + a_2 a_4 + a_s + a_t)} \quad (3.3)$$

becomes relevant as soon as some of the discrete labels a_i are nonzero. The supersymmetric double sine functions $S_\nu(x)$ with $\nu = 0, 1$ are defined in the appendix (A.6).

Before we continue our analysis, let us make a few comments on the status of the definition (3.1), its relation with $\mathrm{U}_q(\mathrm{osp}(1|2))$ and with N=1 Liouville field theory. In recent work, two of the authors and Leszek Hadasz computed the Racah Wigner symbols for a certain series of self-dual representations of the quantum enveloping superalgebra $\mathrm{U}_q(\mathrm{osp}(1|2))$. The arguments of this symbol assume values $\alpha \in Q/2 + i\mathbb{R}$. Furthermore, the symbol defined in [1] was shown to coincide with the fusing matrix of N=1 Liouville field theory when all field labels are taken from the NS sector of the model. The expression in [1] extends the one found by Teschner and Ponsot for $\mathrm{U}_q(\mathrm{sl}(2))$. The latter has been rewritten by Teschner and Vartanov using some highly non-trivial integral identities. Our symbol (3.1) with $a_i = 0$ was defined to extend the Teschner-Vartanov version of the non-supersymmetric symbol to $\mathrm{U}_q(\mathrm{osp}(1|2))$. At the moment we cannot prove that the expression (3.1), $a_i = 0$, agrees with the formula derived in [1] simply because we are missing certain supersymmetric analogues of the integral identities employed in [6]. On the other hand our results below make it seem highly plausible that both formulas agree. In [1] no attempt was made to extend the constructions to the R sector of $N = 1$ Liouville field theory. It is likely that $\mathrm{U}_q(\mathrm{osp}(1|2))$ indeed possesses another self-dual series of representations which can mimic the R sector and that the fusing matrix involving R sector fields may be obtained from the Racah-Wigner symbol in an extended class of self-dual representations, but the details have not been worked out. Here we just make a bold proposal for the extension of the Racah-Wigner symbol to cases with some $a_i \neq 0$. Our results below strongly support a relation with the R sector of N=1 Liouville field theory.

After these comments on the Racah-Wigner symbol (3.1), we would like to repeat the analysis we have performed in section 2. Let us start with the prefactor of our Racah-Wigner symbol. When written in terms of the double sine function, it takes the from

$$\begin{aligned}
\mathcal{P}(\alpha_i, \nu_i) &= \Delta_{\nu_4}(\alpha_s, \alpha_2, \alpha_1)\Delta_{\nu_3}(\alpha_s, \alpha_3, \alpha_4)\Delta_{\nu_2}(\alpha_t, \alpha_3, \alpha_2)\Delta_{\nu_1}(\alpha_4, \alpha_t, \alpha_1) \\
&= (S_{\nu_4+a_s}(\alpha_{12s} - Q)S_{\nu_3+a_s}(\alpha_{s34} - Q)S_{\nu_2+a_t}(\alpha_{23t} - Q)S_{\nu_1+a_t}(\alpha_{14t} - Q))^{\frac{1}{2}} \\
&\quad \Big(S_{\nu_4}(\alpha_{12} - \alpha_s)\, S_{\nu_4+a_1}(\alpha_{1s} - \alpha_2)\, S_{\nu_4+a_2}(\alpha_{2s} - \alpha_1) \\
&\quad S_{\nu_3}(\alpha_{34} - \alpha_s)\, S_{\nu_3+a_4}(\alpha_{s4} - \alpha_3)\, S_{\nu_3+a_3}(\alpha_{3s} - \alpha_4) \\
&\quad S_{\nu_2}(\alpha_{23} - \alpha_t)\, S_{\nu_2+a_2}(\alpha_{t2} - \alpha_3)\, S_{\nu_2+a_3}(\alpha_{3t} - \alpha_2) \\
&\quad S_{\nu_1}(\alpha_{14} - \alpha_t)\, S_{\nu_1+a_1}(\alpha_{1t} - \alpha_4)\, S_{\nu_1+a_4}(\alpha_{4t} - \alpha_1)\Big)^{-\frac{1}{2}} .
\end{aligned} \quad (3.4)$$

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By analogy with the bosonic case we expect that the prefactor vanishes each time one of the external parameters approaches a degenerate value $\alpha_i = -\frac{nb}{2} - \frac{n'}{2b}$ and one of the intermediate parameters $\alpha_x, (x = s, t)$ satisfies the condition

$$\alpha_x = \alpha_j - \frac{xb}{2} - \frac{x'}{2b}, \quad x \in \{-n, -n+2, \dots, n\}, \quad x' \in \{-n', -n'+2, \dots, n'\}, \tag{3.5}$$

where the labels $i, j \in \{1, 2\}$ or $\{3, 4\}$ for $x = s$, and $i, j \in \{2, 3\}$ or $\{1, 4\}$ for $x = t$. Using properties of supersymmetric double sine functions listed in appendix A one can check that the prefactor indeed has zeroes in these cases, provided that the following conditions are satisfied,

$$\begin{aligned} &\frac{n-s}{2} + \frac{n'-s'}{2} \in 2\mathbb{N}+1+\begin{cases} \nu_4, & \text{degenerate } \alpha_i,\ i=1,2 \\ \nu_3, & \text{degenerate } \alpha_i,\ i=3,4 \end{cases} \\ &\frac{n+s}{2} + \frac{n'+s'}{2} \in 2\mathbb{N}+1+\begin{cases} \nu_4 + a_i, & \text{degenerate } \alpha_i,\ i=1,2 \\ \nu_3 + a_i, & \text{degenerate } \alpha_i,\ i=3,4 \end{cases} \end{aligned} \tag{3.6}$$

by the intermediate parameter α_s, and

$$\begin{aligned} &\frac{n-t}{2} + \frac{n'-t'}{2} \in 2\mathbb{N}+1+\begin{cases} \nu_1, & \text{degenerate } \alpha_i,\ i=1,4 \\ \nu_2, & \text{degenerate } \alpha_i,\ i=2,3 \end{cases} \\ &\frac{n+t}{2} + \frac{n'+t'}{2} \in 2\mathbb{N}+1+\begin{cases} \nu_1 + a_i, & \text{degenerate } \alpha_i,\ i=1,4 \\ \nu_2 + a_i, & \text{degenerate } \alpha_i,\ i=2,3 \end{cases} \end{aligned} \tag{3.7}$$

by α_t. As one example, let us discuss the condition (3.6) and suppose that $\alpha_i = \alpha_1 = -\frac{nb}{2} - \frac{n'}{2b}$ for definiteness. It follows that $\alpha_j = \alpha_2$ because α_1 and α_s appear only in combination with α_2 in the arguments of the double sine functions. According to eq. (A.7) the first double sine function $S_{\nu_4}(\alpha_{12} - \alpha_s)$ runs into a pole provided that its argument $\alpha_{12} - \alpha_s = \frac{s-n}{2}b + \frac{s'-n'}{2}b^{-1}$ satisfies $\frac{n-s}{2} + \frac{n'-s'}{2} \in 2\mathbb{N} - 1 + \nu_4$. The second function $S_{\nu_4+a_1}(\alpha_{1s} - \alpha_2)$ has a pole if $\frac{n+s}{2} + \frac{n'+s'}{2} \in 2\mathbb{N} - 1 + \nu_4 + a_1$. If both conditions are fulfilled the prefactor become zero. Let us note that this can be the case only if $s + s' \in 2\mathbb{Z}_{\geq 0} + a_1$ and equivalently, due to eq. (3.5), $n + n' \in 2\mathbb{Z}_{\geq 0} + a_1$. The analysis for the other cases is similar.

In general, the conditions (3.6), (3.7) can be satisfied only if degenerate parameters are of the form

$$\alpha_i = -\frac{nb}{2} - \frac{n'}{2b}, \qquad n + n' \in 2\mathbb{Z}_{\geq 0} + a_i. \tag{3.8}$$

This reflects the situation in the $N = 1$ Liouville field theory, where degenerate representations in the NS and R sectors are labeled by $\alpha_{n,n'}$ with even and odd $n+n'$, respectively. Additionally, the pattern of zeroes of the prefactor $\mathcal{P}(\alpha_i, \nu_i)$ well matches with fusion rules of $N = 1$ Liouville field theory. This provides a first non-trivial test for our proposal.

We plan to test our proposal (3.1) further by continuing it to degenerate parameters, as in the previous section. To this end, let us consider the limit of the Racah-Wigner symbol where $\alpha_2 \to -\frac{nb}{2}$, $\alpha_s \to \alpha_1 - \frac{s}{2}$ and the conditions (3.5)–(3.8) are satisfied. Before

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talking the limit it is useful to pass from the summation over ν to a new summation index $\nu' = \nu + \nu_3 + a_s$. The Racah-Wigner symbol then reads,

$$\begin{Bmatrix} \alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{Bmatrix}^{\nu_3\nu_4}_{\nu_1\nu_2} = \delta_{\sum_i \nu_i = a_s + a_t \, mod\, 2} \, \mathcal{P}(\alpha_i,\nu_i) \int_{\mathcal{C}} du \sum_{\nu'=0}^{1} \Big((-1)^X \, S_{1+\nu_3+\nu_4+\nu'}(u-\alpha_{12s}) \\ S_{1+\nu'}(u-\alpha_{s34}) S_{1+\nu_1+\nu_4+\nu'}(u-\alpha_{23t}) S_{1+\nu_2+\nu_4+\nu'}(u-\alpha_{1t4}) \, S_{\nu_4+\nu'}(\alpha_{1234}-u) \\ S_{\nu_1+\nu'+a_1}(\alpha_{st13}-u) S_{\nu_2+\nu'+a_2}(\alpha_{st24}-u) \, S_{\nu_3+\nu'+a_s}(2Q-u) \Big) \; . \tag{3.9}$$

As in the previous section, we need to determine the singular contributions from the integral

$$\int_{\mathcal{C}'} du \sum_{\nu'=0}^{1} \Big((-1)^X \, S_{1+\nu_3+\nu_4+\nu'}(u-\alpha_{12s}) S_{1+\nu'}(u-\alpha_{s34}) S_{1+\nu_1+\nu_4+\nu'}(u-\alpha_{23t}) \\ S_{1+\nu_2+\nu_4+\nu'}(u-\alpha_{1t4}) \, S_{\nu_4+\nu'}(\alpha_{1234}-u) S_{\nu_1+\nu'+a_1}(\alpha_{st13}-u) S_{\nu_2+\nu'+a_2}(\alpha_{st24}-u) \\ S_{\nu_3+\nu'+a_s}(2Q-u) \Big) = I'_{reg} + I'_1 + I'_2 \; .$$

Note that the product $S_{1+\nu'}(u-\alpha_{s34}) S_{\nu_4+\nu'}(\alpha_{1234}-u)$ has poles in the positions $u = \alpha_{134} - \frac{sb}{2} - pb$ for $p \in \{\nu', \nu'+2, \ldots, \frac{n-s}{2} - \nu'\}$ (ν' keeps track of the parity of p). Due to the "pinching mechanism" each pole contributes a singular term. Once we include the summation over $\nu' = 0, 1$, the sum of singular terms runs through all values of $p \in \{0, 1, \ldots, \frac{n-s}{2}\}$,

$$I'_1 = \sum_{p=0}^{\frac{n-s}{2}} (-1)^X \frac{\left(2\cos(\frac{\pi b^2}{2})\right)^{\frac{s-n}{2}} S_1(0)}{[p]_b! \left[\frac{n-s}{2} - p\right]_b!} S_{1+\nu_3+\nu_4+\nu'}\left(\alpha_{34} - \alpha_1 + \frac{nb}{2} - pb\right) \\ S_{1+\nu_1+\nu_4+\nu'}\left(\alpha_{14} - \alpha_t + \frac{(n-s)b}{2} - pb\right) S_{1+\nu_2+\nu_4+\nu'}\left(\alpha_3 - \alpha_t - \frac{sb}{2} - pb\right) \\ S_{\nu_1+\nu'+a_1}(\alpha_{1t} - \alpha_4 + pb) S_{\nu_2+\nu'+a_2}\left(\alpha_t - \alpha_3 - \frac{nb}{2} + pb\right) S_{\nu_3+\nu'+a_s}\left(2Q - \alpha_{134} + \frac{sb}{2} + pb\right),$$

where we used the shift relations for the supersymmetric double sine function (A.9) and the notation

$$[n]_b! = \begin{cases} \prod_{j=1\, mod\, 2}^{n-1} \cos\left(j\frac{\pi b^2}{2}\right) \prod_{j=2\, mod\, 2}^{n} \sin\left(-j\frac{\pi b^2}{2}\right) \left(\cos\left(\frac{\pi b^2}{2}\right)\right)^{-n}, & \text{for } n \in 2\mathbb{N} \\ \prod_{j=1\, mod\, 2}^{n} \cos\left(j\frac{\pi b^2}{2}\right) \prod_{j=2\, mod\, 2}^{n-1} \sin\left(-j\frac{\pi b^2}{2}\right) \left(\cos\left(\frac{\pi b^2}{2}\right)\right)^{-n}, & \text{for } n \in 2\mathbb{N}+1 \, . \end{cases} \tag{3.10}$$

With the help of conditions (3.6) one can verify that the functions $S_{1+\nu_2+\nu_4+\nu'}(u-\alpha_{1t4})$ $S_{\nu_2+\nu'+a_2}(\alpha_{st24}-u)$ have poles located in $u = \alpha_{1t4} - p'b$, where $p' \in \{\mu, \mu+2, \ldots, \frac{n+s}{2} - \mu\}$,

$\mu = \nu_2 + \nu_4 + \nu' \bmod 2$. They lead to the second sum of singular terms I_2,

$$I_2' = \sum_{p'=0}^{\frac{n+s}{2}} \frac{(-1)^X \left(2\cos\left(\frac{\pi b^2}{2}\right)\right)^{-\frac{n+s}{2}} S_1(0)}{[p']_b! \left[\frac{n+s}{2} - p'\right]_b!} S_{\nu_4+\nu'}\left(\alpha_3 - \alpha_t - \frac{nb}{2} + p'b\right)$$
$$S_{1+\nu_1+\nu_4+\nu'}\left(\alpha_{14} - \alpha_3 + \frac{nb}{2} - p'b\right) S_{1+\nu'}\left(\alpha_t - \alpha_3 + \frac{sb}{2} - p'b\right) S_{\nu_1+\nu'+a_1}\left(\alpha_{13} - \alpha_4 + p'b - \frac{sb}{2}\right)$$
$$S_{1+\nu_3+\nu_4+\nu'}\left(\alpha_{t4} - \alpha_1 + \frac{(n+s)b}{2} - p'b\right) S_{\nu_3+\nu'+a_s}(2Q - \alpha_{1t4} + p'b)\,.$$

Once the two singular contributions from the integral are multiplied by the vanishing prefactor, they give a finite result for the limit of the Racah-Wigner symbol,

$$\lim_{\substack{\alpha_2 \to -\frac{nb}{2} \\ \alpha_s \to \alpha_1 - \frac{sb}{2}}} \begin{Bmatrix} \alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{Bmatrix}_{\nu_1\nu_2}^{\nu_3\nu_4} = \delta_{\sum_i \nu_i = a_s + a_t \bmod 2} \lim_{\substack{\alpha_2 \to -\frac{nb}{2} \\ \alpha_s \to \alpha_1 - \frac{sb}{2}}} \mathcal{P}(\alpha_i, \nu_i)\,(I_1' + I_2')\,. \tag{3.11}$$

The limit above, similar as in the bosonic case (2.10), has simple poles when the second intermediate parameter $\alpha_t \to \alpha_3 - \frac{tb}{2}$ satisfies the conditions (3.5), (3.7). The residue is given by the following formula,

$$\underset{\alpha_t \to \alpha_3 - \frac{tb}{2}}{Res} \left(\lim_{\substack{\alpha_2 \to -\frac{nb}{2} \\ \alpha_s \to \alpha_1 - \frac{sb}{2}}} \begin{Bmatrix} \alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{Bmatrix}_{\nu_1\nu_2}^{\nu_3\nu_4} \right) = \delta_{\sum_i \nu_i = a_s + a_t \bmod 2} \tag{3.12}$$

$$2 \left(\frac{S_{\nu_4+a_s}\left(2\alpha_1 - \frac{(s+n)b}{2} - Q\right) S_{\nu_2+a_t}(2\alpha_3 - \frac{(t+n)b}{2} - Q)}{S_{\nu_4+a_1}\left(2\alpha_1 + \frac{(n-s)b}{2}\right) S_{\nu_2+a_3}(2\alpha_3 + \frac{(n-t)b}{2})} \right)^{\frac{1}{2}}$$
$$\sum_{p=\max\left\{0, \frac{t-s}{2}\right\}}^{\min\left\{\frac{n-s}{2}, \frac{n+t}{2}\right\}} \Bigg\{ (-1)^X \frac{\left(S_{\nu_3+a_s}\left(\alpha_{134} - \frac{sb}{2} - Q\right) S_{\nu_1+a_t}\left(\alpha_{134} - \frac{tb}{2} - Q\right)\right)^{\frac{1}{2}}}{S_{\nu_3+\nu'+a_s}\left(\alpha_{134} - \frac{sb}{2} - pb - Q\right)}$$
$$\frac{\left(\left[\frac{n-s}{2}\right]_b! \left[\frac{n+s}{2}\right]_b! \left[\frac{n-t}{2}\right]_b! \left[\frac{n+t}{2}\right]_b!\right)^{\frac{1}{2}}}{[p]_b! \left[\frac{n-s}{2} - p\right]_b! \left[p + \frac{s-t}{2}\right]_b! \left[\frac{t+n}{2} - p\right]_b!} \frac{S_{\nu_1+\nu'+a_1}(\alpha_{13} - \alpha_4 + pb - \frac{tb}{2})}{(S_{\nu_3+a_3}(\alpha_{13} - \alpha_4 - \frac{sb}{2}) S_{\nu_1+a_1}(\alpha_{13} - \alpha_4 - \frac{tb}{2}))^{\frac{1}{2}}}$$
$$\frac{S_{1+\nu_3+\nu_4+\nu'}\left(\alpha_{34} - \alpha_1 - pb + \frac{nb}{2}\right)}{\left(S_{\nu_3}\left(\alpha_{34} - \alpha_1 + \frac{sb}{2}\right) S_{\nu_1+a_4}\left(\alpha_{34} - \alpha_1 - \frac{tb}{2}\right)\right)^{\frac{1}{2}}} \frac{S_{1+\nu_1+\nu_4+\nu'}\left(\alpha_{14} - \alpha_3 - pb + \frac{(n+t-s)b}{2}\right)}{\left(S_{\nu_3+a_4}\left(\alpha_{14} - \alpha_3 - \frac{sb}{2}\right) S_{\nu_1}\left(\alpha_{14} - \alpha_3 + \frac{tb}{2}\right)\right)^{\frac{1}{2}}} \Bigg\}.$$

In complete analogy to the bosonic case, see eq. (2.12), we shall denote the residue by

$$\begin{Bmatrix} \alpha_1^{a_1} & \alpha_3^{a_3} & (\alpha_1 - \frac{sb}{2})^{a_s} \\ -\frac{nb}{2} & \alpha_4^{a_4} & (\alpha_3 - \frac{tb}{2})^{a_t} \end{Bmatrix}_{\nu_1\nu_2}^{\prime\ \nu_3\nu_4} \equiv \underset{\alpha_t \to \alpha_3 - \frac{tb}{2}}{Res} \left(\lim_{\substack{\alpha_2 \to -\frac{nb}{2} \\ \alpha_s \to \alpha_1 - \frac{sb}{2}}} \begin{Bmatrix} \alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{Bmatrix}_{\nu_1\nu_2}^{\nu_3\nu_4} \right), \tag{3.13}$$

where we assume $n \in 2\mathbb{Z}_{\geq 0} + a_2$, according to the condition (3.8). Now we can send all the other external parameters to degenerate values,

$$\alpha_i \to -j_i b, \qquad 2j_i \in 2\mathbb{Z}_{\geq 0} + a_i.$$

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Using the shift relations (A.9) for double sine functions one obtains

$$\begin{Bmatrix} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{Bmatrix}'^{\,\nu_3\nu_4}_{\nu_1\nu_2} = \delta_{\sum_i \nu_i = 2(j_s+j_t)\, mod\, 2} \frac{(-1)^{A(j_i)}}{2\cos\left(\frac{\pi b^2}{2}\right)\cos\left(\frac{\pi}{2b^2}\right)} \tag{3.14}$$

$$\sum_{z\geq 0} \frac{(-1)^X(-1)^{\frac{1}{2}z(z-1)}\,[z+1]_b!\,\Delta_b(j_s,j_2,j_1)\Delta_b(j_s,j_3,j_4)\Delta_b(j_t,j_3,j_2)\Delta_b(j_4,j_t,j_1)}{[z-j_{12s}]_b!\;[z-j_{34s}]_b!\;[z-j_{14t}]_b!\,[z-j_{23t}]_b!\,[j_{1234}-z]_b!\;[j_{13st}-z]_b!\;[j_{24st}-z]_b!}$$

where we denoted $\frac{n}{2} = j_2$, $\frac{s}{2} = j_s - j_1$, $\frac{t}{2} = j_t - j_3$ and besides conditions (3.6), (3.7) we assume additionally

$$j_{1234} \in 2\mathbb{N} + \nu_3 + \nu_4 + a_s, \quad \text{and} \quad j_{1234} \in 2\mathbb{N} + \nu_1 + \nu_2 + a_t. \tag{3.15}$$

The sum in (3.14) runs over $z = p + j_{s34}$ such that all arguments $[.]_b$ are non-negative, and

$$\Delta_b(a,b,c) = \sqrt{[-a+b+c]_b!\;[a-b+c]_b!\;[a+b-c]_b!/\,[a+b+c+1]_b!}\,.$$

The sign $(-1)^{A(j_i)}$ in the prefactor comes from the identity (A.9) applied to the terms $S_\nu(-xb-Q)$,

$$(-1)^{A(j_i)} = (-1)^{\frac{1}{4}j_{12s}(j_{12s}-1)+\frac{1}{4}j_{s34}(j_{s34}-1)+\frac{1}{4}j_{23t}(j_{23t}-1)+\frac{1}{4}j_{14t}(j_{14t}-1)+1}\,.$$

This concludes our computation of the Racah-Wigner symbol (3.1) for degenerate labels $\alpha_i \to -j_i b$, $2j_i \in 2\mathbb{Z}_{\geq 0} + a_i$.

Let us finally mention that along the same lines one can calculate more general limit of the Racah-Wigner symbol where the parameters take degenerate values,

$$\alpha_i \to -j_i b - j'_i b^{-1}, \qquad j_i + j'_i \in \mathbb{Z}_{\geq 0} + \frac{a_i}{2} \tag{3.16}$$

and the relations (3.6), (3.7) and

$$j_{1234} + j'_{1234} \in 2\mathbb{Z}_{\geq 0} + \nu_3 + \nu_4 + a_s, \quad \text{and} \quad j_{1234} + j'_{1234} \in 2\mathbb{Z}_{\geq 0} + \nu_1 + \nu_2 + a_t \tag{3.17}$$

are assumed. The limit is defined analogously to eqs. (3.13) and (2.17),

$$\begin{Bmatrix} -j_1 b - j'_1 b^{-1} & -j_3 b - j'_3 b^{-1} & -j_s b - j'_s b^{-1} \\ -j_2 b - j'_2 b^{-1} & -j_4 b - j'_4 b^{-1} & -j_t b - j'_t b^{-1} \end{Bmatrix}'^{\,\nu_3\nu_4}_{\nu_1\nu_2}$$

$$\equiv \lim_{\substack{\alpha_j \to -j_j b - j'_j b^{-1} \\ \alpha_k \to -j_k b - j'_k b^{-1} \\ \alpha_l \to -j_l b - j'_l b^{-1}}} \;\underset{\alpha_t \to \alpha_j - \frac{t}{2}b - \frac{t'}{2}b^{-1}}{Res} \left(\lim_{\substack{\alpha_i \to -j_i b - j'_i b^{-1} \\ \alpha_s \to \alpha_k - \frac{s}{2}b - \frac{s'}{2}b^{-1}}} \begin{Bmatrix} \alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{Bmatrix}^{\nu_3\nu_4}_{\nu_1\nu_2} \right),$$

where

$$j_s = j_k + \frac{s}{2}, \quad j'_s = j'_k + \frac{s'}{2}; \qquad j_t = j_j + \frac{t}{2}, \quad j'_t = j'_j + \frac{t'}{2}.$$

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Using the identity (A.11) for double sine functions $S_\nu(-xb-yb^{-1})$ one may obtain

$$\left\{ \begin{matrix} -j_1b-j_1'b^{-1} & -j_3b-j_3'b^{-1} & -j_sb-j_s'b^{-1} \\ -j_2b-j_2'b^{-1} & -j_4b-j_4'b^{-1} & -j_tb-j_t'b^{-1} \end{matrix} \right\}'^{\,\nu_3\nu_4}_{\nu_1\nu_2} \sim \delta_{\sum_i \nu_i = 2(j_s+j_t)\,mod\,2}$$
$$\sum_{z\geq 0}\sum_{z'\geq 0}(-1)^X(-1)^{\frac12 z(z-1)+\frac12 z'(z'-1)}(-1)^B[z+1]_b!\,[z'+1]_{\frac1b}!$$
$$\Big([z-j_{12s}]_b!\,[z-j_{34s}]_b![z-j_{14t}]_b!\,[z-j_{23t}]_b![j_{1234}-z]_b!\Big)^{-1} \tag{3.18}$$
$$\Big([j_{13st}-z]_b!\,[j_{24st}-z]_b!\,[z'-j_{12s}']_{\frac1b}!\,[z'-j_{34s}']_{\frac1b}!\,[z'-j_{14t}']_{\frac1b}!\Big)^{-1}$$
$$\Big(\,[z'-j_{23t}']_{\frac1b}!\,[j_{1234}'-z']_{\frac1b}!\,[j_{13st}'-z']_{\frac1b}!\,[j_{24st}'-z']_{\frac1b}!\Big)^{-1}\,.$$

The result is similar to eq. (3.14), with the difference that now we have two sets of brackets $[x]_b\,,[y]_{\frac1b}$ defined by the formula (3.10) and the analogous one with b exchanged for b^{-1}. Moreover an additional sign comes from eq. (A.11),

$$(-1)^B = (-1)^{-2zj_{1234st}'-2z'j_{1234st}}(-1)^{\frac12\sum_{i=1}^7\mu_i(z-x_i)^2-\nu(\frac{z^2}{2}-z)} \tag{3.19}$$

where

$$\begin{aligned} \mu_1 &= 1+\nu+\nu_4+a_s \text{ mod } 2, & x_1 &= j_{12s}; & \mu_5 &= \nu+\nu_1+\nu_2+a_t \text{ mod } 2, & x_5 &= j_{1234};\\ \mu_2 &= 1+\nu+\nu_3+a_s \text{ mod } 2, & x_2 &= j_{s34}; & \mu_6 &= \nu+\nu_1+\nu_3+a_2 \text{ mod } 2, & x_6 &= j_{st13};\\ \mu_3 &= 1+\nu+\nu_2+a_t \text{ mod } 2, & x_3 &= j_{23t}; & \mu_7 &= \nu+\nu_1+\nu_4+a_3 \text{ mod } 2, & x_7 &= j_{st24};\\ \mu_4 &= 1+\nu+\nu_1+a_t \text{ mod } 2, & x_4 &= j_{1t4}; \end{aligned}$$

The final formulas (3.14), (3.18) look somewhat similar to the corresponding equations in section 2. We are now going to see that they are indeed very closely related.

4 Comparison with the finite dimensional 6J symbols

Our formulas (3.14), (3.18) for the limiting value of the proposed Racah-Wigner symbol could turn into a strong test of eq. (3.1) provided we were able to show that the expressions (3.14), (3.18) give rise to a solution of the pentagon equation. In our discussion of the Racah-Wigner symbol for $U_q(sl(2))$ this followed from the comparison with the 6J symbols for finite dimensional representations. By construction, the latter are known to satisfy the pentagon equation. By analogy one might now hope that the coefficients (3.14), (3.18) coincide with the 6J symbols for finite dimensional representations of the quantum universal enveloping algebra $U_q(osp(1|2))$. This, however, is not quite the case. To start the comparison, we quote an expression for the 6J symbols of $U_q(osp(1|2))$ from [9, 10],

$$\begin{bmatrix} l_1 & l_2 & l_s \\ l_3 & l_4 & l_t \end{bmatrix}_q = (-1)^{\frac12(l_{1234}+l_s+l_t)(l_{1234}+l_s+l_t+1)+\frac12\left(\sum_{i=1}^4 l_i(l_i-1)+l_s(l_s-1)+l_t(l_t-1)\right)} \tag{4.1}$$
$$\sum_{z\geq 0}\frac{(-1)^{\frac12 z(z-1)}[z+1]_q'!\Delta_q'(l_s,l_2,l_1)\Delta_q'(l_s,l_3,l_4)\Delta_q'(l_t,l_3,l_2)\Delta_q'(l_4,l_t,l_1)}{[z-l_{12s}]_q'!\,[z-l_{34s}]_q'!\,[z-l_{14t}]_q'![z-l_{23t}]_q'![l_{1234}-z]_q'!\,[l_{13st}-z]_q'!\,[l_{24st}-z]_q'!}$$

where the sum extend over those values of z for which all arguments of the quantum number $[.]'_q$ are non-negative and

$$\Delta'_q(a,b,c) = \sqrt{[-a+b+c]'_q!\,[a-b+c]'_q!\,[a+b-c]'_q!/[a+b+c+1]'_q!}\,.$$

Let us stress that irreducible finite dimensional representations of $\mathrm{U}_q(\mathrm{osp}(1|2))$ are labeled by integers l. Hence all the arguments l_i in the above 6J symbols satisfy $l_i \in \mathbb{Z}_{\geq}0$. In the previous definition the q-number $[.]'_q$ is defined as

$$[n]'_q = \frac{q^{-\frac{n}{2}} - (-1)^n q^{\frac{n}{2}}}{q^{-\frac{1}{2}} + q^{\frac{1}{2}}}\ . \tag{4.2}$$

For $q = e^{i\pi b^2}$ the quantum factorial takes the form

$$[n]'_q! = \begin{cases} \prod_{j=1\,mod\,2}^{n-1} \cos\left(j\frac{\pi b^2}{2}\right) \prod_{j=2\,mod\,2}^{n}\left(i\sin\left(-j\frac{\pi b^2}{2}\right)\right)\left(\cos\left(\frac{\pi b^2}{2}\right)\right)^{-n}, & \text{for}\, n \in 2\mathbb{N} \\ \prod_{j=1\,mod\,2}^{n} \cos\left(j\frac{\pi b^2}{2}\right) \prod_{j=2\,mod\,2}^{n-1}\left(i\sin\left(-j\frac{\pi b^2}{2}\right)\right)\left(\cos\left(\frac{\pi b^2}{2}\right)\right)^{-n}, & \text{for}\, n \in 2\mathbb{N}+1. \end{cases}$$

It is related to the similar symbol $[.]_b!$ which we defined in eq. (3.10) through

$$[n]_b! = (-1)^{\frac{1}{12}n(n+1)(2n+1)}(-i)^n\ [n]'_q!\ . \tag{4.3}$$

In order to compare the limiting values (3.14) of Racah-Wigner symbols (3.1) with the 6J symbols (4.1) we rewrite the latter in terms of the new symbol $[n]'_q$,

$$\begin{aligned} \left\{\begin{matrix} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{matrix}\right\}'^{\,\nu_3\nu_4}_{\nu_1\nu_2} &= \delta_{\sum_i \nu_i = 2(j_s+j_t)\,mod\,2} \frac{(-1)^{A'(j_i)}\Delta'_q(j_s,j_2,j_1)\Delta'_q(j_s,j_3,j_4)}{2\cos\left(\frac{\pi b^2}{2}\right)\cos\left(\frac{\pi}{2b^2}\right)} \\ &\Delta'_q(j_t,j_3,j_2)\Delta'_q(j_4,j_t,j_1)\sum_{z\geq 0}(-1)^X(-1)^{\frac{1}{2}z(z-1)+2z(j_{1234st}+j_1j_3+j_2j_4+j_sj_t)}[z+1]'_q! \\ &\Big([z-j_{12s}]'_q!\,[z-j_{34s}]'_q![z-j_{14t}]'_q!\,[z-j_{23t}]'_q![j_{1234}-z]'_q!\,[j_{13st}-z]'_q!\,[j_{24st}-z]'_q!\Big)^{-1} \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} (-1)^{A'(j_i)} &= (-1)^{-\frac{1}{2}-(j_{1234st}+1)(j_1j_3+j_2j_4+j_sj_t+1)+\frac{1}{2}j_{12s}(j_{12s}-1)+\frac{1}{2}j_{s34}(j_{s34}-1)} \\ &\quad (-1)^{\frac{1}{2}j_{23t}(j_{23t}-1)+\frac{1}{2}j_{14t}(j_{14t}-1)-F(j_1,j_2,j_s)-F(j_3,j_4,j_s)-F(j_2,j_3,j_t)-F(j_1,j_4,j_t)}\,, \\ (-1)^{F(j_1,j_2,j_3)} &= (-1)^{\frac{3}{4}j_{123}(j_{123}+1)+j_1j_2j_3+j_1j_2+j_1j_3+j_2j_3}\ . \end{aligned}$$

In the case when all j_i are integer, or equivalently all $a_i = 0$, the sign $(-1)^X$ defined in eq. (3.3) and $(-1)^{2z(j_{1234st}+j_1j_3+j_2j_4+j_sj_t)}$ both vanish so that we can relate the limit of the Racah-Wigner symbol (4.4) to the $\mathrm{U}_q(\mathrm{osp}(1|2))$ 6J coefficients (4.1),

$$\left\{\begin{matrix} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{matrix}\right\}'^{\,\nu_3\nu_4}_{\nu_1\nu_2} = \delta_{\sum_i \nu_i = 2(j_s+j_t)\,mod\,2} \frac{(-1)^{A''(j_i)}}{2\cos\left(\frac{\pi b^2}{2}\right)\cos\left(\frac{\pi}{2b^2}\right)} \begin{bmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{bmatrix}_q \tag{4.5}$$

where

$$(-1)^{A''(j_i)} = (-1)^{\frac{1}{2}-j_{1234st}(j_1j_3+j_2j_4+j_sj_t)-F(j_1,j_2,j_s)-F(j_3,j_4,j_s)-F(j_2,j_3,j_t)-F(j_1,j_4,j_t)} .$$

Let us emphasize that in arriving at the expressions (3.14) for the limiting values of the Racah-Wigner symbol, the parameters j_i were allowed to take either integer ($a_i = 0$) or half-integer ($a_i = 1$) values. We have now shown that the limit is proportional to the $\mathrm{U}_q(\mathrm{osp}(1|2))$ 6J coefficients, provided all arguments j_i are integer. In order to find an interpretation of the limit (3.14) in the case of half-integer j_i, we will have to bring in a different idea. It is related to an intriguing duality between the 6J symbol of $\mathrm{U}_q(\mathrm{osp}(1|2))$ and $\mathrm{U}_q(\mathrm{sl}(2))$.

As was originally noticed in [11, 12], the $\mathrm{U}_q(\mathrm{sl}(2))$ quantum numbers (2.14) with the deformation parameter $q' = i\sqrt{q}$ are related to the $\mathrm{U}_q(\mathrm{osp}(1|2))$ quantum numbers (4.2) through,

$$[x]_{q'} = (-1)^{\frac{1-x}{2}}\, [x]'_q \ . \tag{4.6}$$

This equation implies a relation between the quantum factorials,

$$[x]'_q! = (-1)^{\frac{x(x-1)}{4}}\, [x]_{q'}! \ . \tag{4.7}$$

With its help we can rewrite the $\mathrm{U}_q(\mathrm{osp}(1|2))$ 6J symbol in terms of the $\mathrm{U}_q(\mathrm{sl}(2))$ quantum factorials,

$$\begin{bmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{bmatrix}_q = (-1)^{\sum_{i=1}^4 \frac{j_i}{2}(j_i-1)+\frac{j_s}{2}(j_s-1)+\frac{j_t}{2}(j_t-1)-\frac{1}{2}j_{st}j_{1234}-\frac{1}{2}j_{13}j_{24}}$$
$$\sum_{z\geq 0}\frac{(-1)^{z+2zj_{1234st}}[z+1]_{q'}!\Delta_{q'}(j_s,j_2,j_1)\Delta_{q'}(j_s,j_3,j_4)\Delta_{q'}(j_t,j_3,j_2)\Delta_{q'}(j_4,j_t,j_1)}{[z-j_{12s}]_{q'}!\,[z-j_{34s}]_{q'}!\,[z-j_{14t}]_{q'}![z-j_{23t}]_{q'}![j_{1234}-z]_{q'}!\,[j_{13st}-z]_{q'}!\,[j_{24st}-z]_{q'}!}.$$

Due to the condition $j_i \in \mathbb{Z}_{\geq 0}$ in the $\mathrm{U}_q(\mathrm{osp}(1|2))$ 6J symbol, the sign $(-1)^{2zj_{1234st}}$ vanishes and one arrives at the following relation between the 6J symbols (4.1) and (2.15)

$$\begin{bmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{bmatrix}_q = (-1)^{\sum_{i=1}^4 \frac{j_i}{2}(j_i-1)+\frac{j_s}{2}(j_s-1)+\frac{j_t}{2}(j_t-1)-\frac{1}{2}j_{st}j_{1234}-\frac{1}{2}j_{13}j_{24}}$$
$$\frac{(-1)^{-j_{12}+j_{34}+2j_s}}{\sqrt{[2j_s+1]_{q'}[2j_t+1]_{q'}}}\begin{pmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{pmatrix}_{q'} .$$

In a similar way we can relate our limit of Racah-Wigner coefficients (4.4) to the 6J symbol of $\mathrm{U}_{q'}(\mathrm{sl}(2))$ even if some of the arguments j_i assume (half-)integer values. When written in terms of $[x]_{q'}$, the Racah-Wigner coefficients (4.4) take the following form,

$$\begin{Bmatrix} -j_1b & -j_3b & -j_sb \\ -j_2b & -j_4b & -j_tb \end{Bmatrix}'^{\,\nu_3\nu_4}_{\nu_1\nu_2} = \delta_{\sum_i \nu_i = 2(j_s+j_t)\,\mathrm{mod}\,2}\frac{(-1)^{A'''(j_i)}\Delta_{q'}(j_s,j_2,j_1)\Delta_{q'}(j_s,j_3,j_4)}{2\cos\left(\frac{\pi b^2}{2}\right)\cos\left(\frac{\pi}{2b^2}\right)}$$
$$\Delta_{q'}(j_t,j_3,j_2)\Delta_{q'}(j_4,j_t,j_1)\sum_{z\geq 0}(-1)^X(-1)^{z+2(z+1)(j_1j_3+j_2j_4+j_sj_t)}[z+1]_{q'}! \tag{4.8}$$
$$\Big([z-j_{12s}]_{q'}!\,[z-j_{34s}]_{q'}![z-j_{14t}]_{q'}!\,[z-j_{23t}]_{q'}![j_{1234}-z]_{q'}!\,[j_{13st}-z]_{q'}!\,[j_{24st}-z]_{q'}!\Big)^{-1}$$

where

$$(-1)^{A'''(j_i)} = (-1)^{\frac{1}{2}-(j_{1234st}+2)(j_1j_3+j_2j_4+j_sj_t)-F'(j_1,j_2,j_s)-F'(j_3,j_4,j_s)-F'(j_2,j_3,j_t)-F'(j_1,j_4,j_t)},$$
$$(-1)^{F'(j_1,j_2,j_3)} = (-1)^{j_1j_2j_3+\frac{1}{2}(j_1+j_2+j_3)}.$$

Using the relations (3.6), (3.7) and (3.15) one may check that

$$(-1)^{2j_1j_3+2j_2j_4+2j_sj_t} = (-1)^{a_s\nu_1+a_1\nu_3+a_4\nu_4+a_1a_s+a_2a_4+a_s+a_t}. \tag{4.9}$$

Since the parameter z is related to the summation parameter p (3.12) as $z = p + j_{34s}$ and the parity of p is tracked by $\nu' = \nu + \nu_3 + a_s$, we may relate the sign under the sum in eq. (4.8) to the sign factor $(-1)^X$ that was defined in eq. (3.3),

$$\begin{aligned}(-1)^{2(z+1)(j_1j_3+j_2j_4+j_sj_t)} &= (-1)^{2(\nu+\nu_3+a_s+j_{34s}+1)(j_1j_3+j_2j_4+j_sj_t)} \\ &= (-1)^{\nu(a_s\nu_1+a_1\nu_3+a_4\nu_4+a_1a_s+a_2a_4+a_s+a_t)} = (-1)^X,\end{aligned} \tag{4.10}$$

where we used eq. (3.15) to check that $\nu + \nu_3 + a_s + j_{34s} + 1 \in 2\mathbb{N} + 2(\nu + \nu_3 + \nu_4 + a_s) + \nu$. Thus the limit (4.8) is proportional to the 6J symbol of finite dimensional representations of $\mathrm{U}_{q'}(\mathrm{sl}(2))$,

$$\begin{Bmatrix} -j_1b & -j_3b & -j_sb \\ -j_2b & -j_4b & -j_tb \end{Bmatrix}'^{\,\nu_3\nu_4}_{\nu_1\nu_2} = \delta_{\sum_i \nu_i = 2j_s+2j_t\, mod\, 2}\, \frac{(-1)^{A'''(j_i)}}{2\cos\left(\frac{\pi b^2}{2}\right)\cos\left(\frac{\pi}{2b^2}\right)} \frac{(-1)^{-j_{12}+j_{34}+2j_s}}{\sqrt{[2j_s+1]_{q'}[2j_t+1]_{q'}}} \begin{pmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{pmatrix}_{q'}. \tag{4.11}$$

This concludes our discussion of the limiting Racah-Wigner coefficients (3.14). Our analysis has shown that the expression we obtained from our proposal (3.1) is dual to the 6J symbol for finite dimensional representations of the quantum universal enveloping algebra $\mathrm{U}_q(\mathrm{sl}(2))$. By construction the latter satisfy the pentagon equation. Even though we have not demonstrated that the original symbol (3.1) solved the pentagon identity for arbitrary values of the weights α, our results provide highly non-trivial evidence in favor of the proposal. Note in particular that our sign factors were rather crucial in making things work as soon as some of the parameters had non-zero label a_i, what corresponds to R sector of $N = 1$ Liouville field theory.

It is actually possible to carry things a bit further. As we noted before, the evaluation of the Racah-Wigner symbol (3.1) is possible for general degenerate parameters. In that case, the limiting values of the Racah-Wigner symbol (3.18) can be also related to $\mathrm{U}_q(\mathrm{sl}(2))$ 6J symbols,

$$\begin{Bmatrix} -j_1b - j_1'b^{-1} & -j_3b - j_3'b^{-1} & -j_sb - j_s'b^{-1} \\ -j_2b - j_2'b^{-1} & -j_4b - j_4'b^{-1} & -j_tb - j_t'b^{-1} \end{Bmatrix}'^{\,\nu_3\nu_4}_{\nu_1\nu_2} \sim \delta_{\sum_i \nu_i = 2j_s+2j_t\, mod\, 2} \begin{pmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{pmatrix}_{q'} \begin{pmatrix} j_1' & j_2' & j_s' \\ j_3' & j_4' & j_t' \end{pmatrix}_{q''}, \tag{4.12}$$

where the deformation parameters are $q'^2 = -q = e^{i\pi(b^2-1)}$ and $q''^2 = e^{i\pi(b^{-2}-1)}$. The above factorization occurs when the sign $(-1)^X$ defined by eq. (3.3) cancels the factor $(-1)^B$ from eq. (3.19) multiplied by the sign in eq. (4.8) and the corresponding one depending on j_i', i.e. whenever

$$(-1)^X(-1)^B(-1)^{2z(j_1j_3+j_2j_4+j_sj_t)+2z'(j_1'j_3'+j_2'j_4'+j_s'j_t')} = 1 \; .$$

We verified this relation for degenerate parameters $\alpha_i = -j_ib - j_i'b^{-1}$ with $a_i = 0$ satisfying $j_i - j_i' \in 2\mathbb{Z}$ and for arbitrary degenerate parameters with $a_i = 1$.

As in the bosonic case (2.18), we can relate our result with the fusion matrix of supersymmetric minimal models. The degenerate representations of NSR algebra are parametrized by a pair of Kac labels $(2j, 2j')$, satisfying $j_i + j_i' \in \mathbb{Z}_{\geq 0}$, $j_i - j_i' \in 2\mathbb{Z}$ in the NS sector and $j_i + j_i' \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ in the R sector. It follows from the coset construction

$$\mathrm{SMM}_k = (\mathrm{SU}(2)_k \times \mathrm{SU}(2)_2)/\mathrm{SU}(2)_{k+2}$$

of supersymmetric minimal models that the fusion matrix is given in terms of two 6J symbols of $\mathrm{U}_q(\mathrm{sl}(2))$ with deformation parameters $q_1^2 = \exp(2i\pi/(k+2))$ and $q_2^2 = \exp(2i\pi/(k+4))$. Taking into account the symmetry $q_i \leftrightarrow q_i^{-1}$, these values match perfectly those in the 6J symbols on the right hand side of eq. (4.12) if we set $b^2 = (k+4)/(k+2)$.

With all these non-trivial test being performed, we trust that our formula (3.1) correctly describes the fusing matrix of $N = 1$ Liouville field theory for both NS and R sector fields.

5 Conclusions

In this work we proposed a formula (3.1) for the Racah-Wigner symbol of the non-compact quantum universal enveloping algebra $\mathrm{U}_q(\mathrm{osp}(1|2))$. In order to test our proposal we continued the symbol to a discrete set of parameters $\alpha = -jb - j'b^{-1}$, $j, j' \in \mathbb{Z}_{\geq 0}/2$. For integer $j \in \mathbb{N}$ and $j' = 0$ we recovered the known expressions for Racah-Wigner coefficients of finite dimensional $\mathrm{U}_q(\mathrm{osp}(1|2))$ representations. Half integer values j are not related to the 6J symbols of $\mathrm{U}_q(\mathrm{osp}(1|2))$ but rather to those of $\mathrm{U}_q(\mathrm{sl}(2))$. The relation is furnished by a duality which extends the known correspondence between finite dimensional representations of $\mathrm{U}_q(\mathrm{osp}(1|2))$ and integer spin representations of $\mathrm{U}_q(\mathrm{sl}(2))$ to the case of half-integer spins. A related extension was also uncovered by Mikhaylov and Witten [19]. For cases with $j' \neq 0$ we also discussed the expected relation with the fusing matrix of unitary superconformal minimal models. There are a number of interesting open issues that merit further investigation.

As we stressed before, the Racah-Wigner symbol (3.1) should coincide with the complete fusing matrix of N=1 Liouville field theory in both the NS and the R sector [2, 3, 20]. For NS sector representations a related statement was established in [1]. Of course, it would be interesting to incorporate R sector representations into this comparison. Our comments on the relation with the fusing matrix of minimal models supports such an identification very strongly. Assuming that our Racah-Wigner symbol can be reinterpreted as the fusing matrix in N=1 Liouville theory, our expression (3.1), and special cases thereof, should then also describe various operator product coefficients in the bulk and boundary theory, and in particular the coefficients of boundary operator product expansion, see e.g. [21] for a review of the relation.

Recently, it has been observed that the operator product coefficients of N=1 Liouville field theory with central charge $c = 15/2 + 3(b^2 + b^{-2})$ can be factorized into a products of the coefficients in ordinary (non-supersymmetric) Liouville field theory and those of an imaginary (time-like) version thereof [22–25]. The central charges of the latter are given by $c_i = 13 + 6(b_i^2 + b_i^{-2})$ for $i = 1, 2$ with

$$b_1^2 = \frac{1}{2}(b^2 - 1)\,, \qquad b_2^2 = 2(b^{-2} - 1)^{-1} = -b_1^{-2} - 2\,.$$

This suggest a relation between Racah-Wigner symbols of non-compact $U_q(\mathrm{osp}(1|2))$ for $q = \exp i\pi b^2$ and those of $U_{q_i}(\mathrm{sl}(2))$ for the two values $q_1 = \exp(i\pi b_1^2) = \sqrt{-q}$ and $q_2 = \tilde{q}_1$. Note that the latter is obtained from the former by modular transformation. We see sign of such a relation in the limit of discrete parameters (4.12), where two 6J symbols for finite dimensional representations of $U_q(\mathrm{sl}(2))$ with $q' = e^{i\pi b_1^2}$ and $q'' = e^{i\pi b_2^{-2}}$ occur. We plan to investigate the extension of the duality between $U_q(\mathrm{osp}(1|2))$ and $U_q(\mathrm{sl}(2))$ to the continuous self-dual series of representations in future work. It should also be linked with a strong-weak coupling duality between the non-compact OSP(2|1)/U(1) cigar-like coset model and double Liouville theory that was described in [26].

As we recalled in the introduction, the fusing matrix of $N = 1$ Liouville field theory should be a central ingredient in the construction of a new 3-dimensional topological quantum field theory, just as Faddeev's quantum dilogarithm [27, 28], i.e. the building block of the fusing matrix on Liouville field theory, is used to construct SL(2) Chern-Simons or quantum Teichmueller theory, see e.g. [29–35]. We will explore these aspects of our work in a future publication.

Acknowledgments

We wish to thank Leszek Hadasz, Jörg Teschner, Grigory Vartanov and Edward Witten for discussions and useful comments. This work was supported in part by the GRK 1670 "Mathematics inspired by Quantum Field and String Theory". The work of PS was supported by the Kolumb Programme KOL/6/2011-I of FNP, by the NCN grant DEC2011/01/B/ST1/01302 and by the European Social Fund grant POKL.04.01.01-00-054/10-00.

A Double sine functions

The double sine function $S_b(x)$ is given in terms of Barnes' double Gamma function through

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q - x)} \tag{A.1}$$

and has poles in positions x such that

$$S_b(x)^{-1} = 0 \qquad \Longleftrightarrow \qquad x = -nb - mb^{-1}\,, \qquad n, m \in \mathbb{Z}_{\geq 0}\,. \tag{A.2}$$

It satisfies the shift relations

$$S_b(x + b^{\pm 1}) = 2\sin(\pi b^{\pm 1} x)\, S_b(x)\,, \tag{A.3}$$

which imply that one can evaluate

$$\begin{aligned} S_b(-kb) &= \prod_{j=1}^{k} \left(2\sin(-\pi j b^2)\right)^{-1} S_b(0) = \left(-2\sin(\pi b^2)\right)^{-k} \frac{S_b(0)}{[k]!}\,, \\ S_b(-kb - Q) &= \left(2\sin(\pi b^2)\right)^{-k-1} \left(2\sin(-\pi b^{-2})\right)^{-1} \frac{S_b(0)}{[k+1]!}\,, \end{aligned} \tag{A.4}$$

for $k \in \mathbb{N}$, and more general

$$S_b(-xb - yb^{-1}) = \left(2\sin(\pi b^2)\right)^{-x} \left(2\sin(-\pi b^{-2})\right)^{-y} \frac{(-1)^{xy} S_b(0)}{[x]![y]'!}\,, \tag{A.5}$$

for $x, y \in \mathbb{Z}_{\geq 0}$. We have also used the q-number $[x] = \frac{\sin(\pi b^2 x)}{\sin \pi b^2}$ and $[y]' = \frac{\sin(\pi b^{-2} y)}{\sin \pi b^{-2}}$.

The supersymmetric double sine functions are constructed from Barnes' double Gamma functions

$$\begin{aligned} S_1(x) &= S_{NS}(x) = \frac{\Gamma_b\left(\frac{x}{2}\right)\Gamma_b\left(\frac{x+Q}{2}\right)}{\Gamma_b\left(\frac{Q-x}{2}\right)\Gamma_b\left(\frac{2Q-x}{2}\right)} \\ S_0(x) &= S_R(x) = \frac{\Gamma_b\left(\frac{x+b}{2}\right)\Gamma_b\left(\frac{x+b^{-1}}{2}\right)}{\Gamma_b\left(\frac{Q-x+b}{2}\right)\Gamma_b\left(\frac{Q-x+b^{-1}}{2}\right)} \end{aligned} \tag{A.6}$$

and they have poles as

$$S_\nu(x)^{-1} = 0 \qquad \Longleftrightarrow \qquad x = kb + l/b\,, \quad k, l \in \mathbb{Z}_{\geq 0}\,, \quad k + l \in 2\mathbb{N} - 1 - \nu. \tag{A.7}$$

They obey the shift relations:

$$S_1(x + b^{\pm 1}) = 2\cos\left(\frac{\pi b^{\pm 1} x}{2}\right) S_0(x), \qquad S_0(x + b^{\pm 1}) = 2\sin\left(\frac{\pi b^{\pm 1} x}{2}\right) S_1(x). \tag{A.8}$$

For x integer such that $x \in 2\mathbb{N} - 1 - \nu$ the double sine functions can be written as:

$$S_\nu(-xb) = \frac{S_1(0)}{\left(2\cos\left(\frac{\pi b^2}{2}\right)\right)^x [x]_b!} \tag{A.9}$$

$$S_\nu(-xb - Q) = \frac{(-1)^{-\frac{x+1}{2} - \frac{1}{2}\delta_{\nu,1}}\, S_1(0)}{2\cos\left(\frac{\pi}{2b^2}\right)\left(2\cos\left(\frac{\pi b^2}{2}\right)\right)^{x+1} [x+1]_b!} = \frac{(-1)^{-\frac{x(x-1)}{2} + 1}\, S_1(0)}{2\cos\left(\frac{\pi}{2b^2}\right)\left(2\cos\left(\frac{\pi b^2}{2}\right)\right)^{x+1} [x+1]_b!}$$

where

$$[n]_b! = \begin{cases} \prod_{j=1\, mod\, 2}^{n-1} \cos\left(j\frac{\pi b^2}{2}\right) \prod_{j=2\, mod\, 2}^{n} \sin\left(-j\frac{\pi b^2}{2}\right)\left(\cos\left(\frac{\pi b^2}{2}\right)\right)^{-n}, & \text{for } n \in 2\mathbb{N} \\ \prod_{j=1\, mod\, 2}^{n} \cos\left(j\frac{\pi b^2}{2}\right) \prod_{j=2\, mod\, 2}^{n-1} \sin\left(-j\frac{\pi b^2}{2}\right)\left(\cos\left(\frac{\pi b^2}{2}\right)\right)^{-n}, & \text{for } n \in 2\mathbb{N}+1. \end{cases} \tag{A.10}$$

In general, for arguments such that $x + y \in 2\mathbb{N} - 1 - \nu$, the double sine functions satisfy the identity:

$$S_\nu(-xb - yb^{-1}) = \frac{(-1)^{\frac{xy}{2}+\nu\frac{x^2}{2}}\, S_1(0)}{\left(2\cos\left(\frac{\pi b^2}{2}\right)\right)^x \left(2\cos\left(\frac{\pi}{2b^2}\right)\right)^y\, [x]_b!\, [y]_{\frac{1}{b}}!} \tag{A.11}$$

where $[n]_{\frac{1}{b}}!$ is given by the formula (A.10) with b exchanged for b^{-1}.

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